

# LOCAL WEGNER AND MINAMI ESTIMATES FOR CONTINUOUS RANDOM SCHRÖDINGER OPERATORS

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**ABSTRACT.** We introduce and prove local Wegner estimates for continuous generalized Anderson Hamiltonians, where the single-site random variables are independent but not necessarily identically distributed. We use them to prove Minami's estimate for continuous Anderson Hamiltonians whose single site probability distribution has a bounded density and satisfy a simple covering condition. Eigenvalues statistics such as Poisson statistics and level spacings statistics follow for such continuous Anderson Hamiltonians, as well as simplicity of eigenvalues. As another application of local Wegner estimates, we show that the (differentiated) density of states exhibits the same Lifshitz tails upper bound as the integrated density of states.

## 1. INTRODUCTION

In this paper we introduce the new concept of local Wegner estimates for continuous generalized Anderson Hamiltonians, where the single-site random variables are independent but not necessarily identically distributed. We prove several such local Wegner estimates, and apply them to prove Minami's estimate for continuous Anderson Hamiltonians whose single-site probability distribution has a bounded density and satisfy a simple covering condition, removing the requirements of a uniform-like probability distribution and a double covering condition from our original proof [CoGK2]. Eigenvalues statistics such as Poisson statistics [CoGK2, GK11] and level spacings statistics [GK11] follow for such continuous Anderson Hamiltonians, as well as simplicity of eigenvalues [K1M, CoGK2]. As another application of local Wegner estimates, we show that the (differentiated) density of states exhibits the same Lifshitz tails upper bound as the integrated density of states.

We consider continuous generalized Anderson Hamiltonians, which are random Schrödinger operators on  $L^2(\mathbb{R}^d)$  of the type

$$H_{\omega} := -\Delta + V_{\text{per}} + V_{\omega}, \quad (1.1)$$

where:  $\Delta$  is the  $d$ -dimensional Laplacian operator;  $V_{\text{per}}$  is a bounded  $q\mathbb{Z}^d$ -periodic potential with  $q \in \mathbb{N}$ ; and  $V_{\omega}$  is an alloy-type random potential:

$$V_{\omega}(x) := \sum_{j \in \mathbb{Z}^d} \omega_j u_j(x), \quad \text{with} \quad u_j(x) = u(x - j), \quad (1.2)$$

where the single site potential  $u$  is a nonnegative bounded measurable function on  $\mathbb{R}^d$  with compact support, uniformly bounded away from zero in a neighborhood of the origin, and  $\omega = \{\omega_j\}_{j \in \mathbb{Z}^d}$  is a family of independent (not necessarily identical)

random variables, such that, with  $\mu_j$  denoting the probability distribution of  $\omega_j$ ,

$$\bigcup_{j \in \mathbb{Z}^d} \text{supp } \mu_j \subset [M_-, M_+] \quad \text{for some} \quad -\infty < M_- < M_+ < \infty. \quad (1.3)$$

Moreover, in this paper we always assume that  $\mu_j$  has no atoms for all  $j \in \mathbb{Z}^d$ . If the  $\{\omega_j\}_{j \in \mathbb{Z}^d}$  are also identically distributed, i.e.,  $\mu_j = \mu$  for all  $j \in \mathbb{Z}^d$ ,  $H_\omega$  is the (usual) Anderson Hamiltonian.

Given a finite Borel measure  $\nu$  on  $\mathbb{R}$  and  $s \geq 0$ , we let  $S_\nu(s) := \sup_{a \in \mathbb{R}} \nu([a, a+s])$ , the concentration function of  $\nu$ , and set

$$Q_\nu(s) := \begin{cases} \|\rho\|_\infty s & \text{if } \nu \text{ has a bounded density } \rho \\ 8S_\nu(s) & \text{otherwise} \end{cases}. \quad (1.4)$$

$Q_\nu(s)$  is continuous on  $[0, \infty[$  if and only if the measure  $\nu$  has no atoms, in which case  $\lim_{s \downarrow 0} Q_\nu(s) = Q_\nu(0) = 0$  [HT].

The restriction of  $H_\omega$  to a finite box has a finite number of eigenvalues in a given bounded interval  $I \in \mathbb{R}$ . Fluctuations of these eigenvalues due to the random variables  $\{\omega_j\}_{j \in \mathbb{Z}^d}$  play a crucial role in the understanding of the localization properties of  $H_\omega$ . When averaging over a single random variable, the fluctuations of the eigenvalues are controlled thanks to a spectral averaging principle: given a trace class operator  $S \geq 0$ , we have [CoH, CoHK2],

$$\mathbb{E}_{\omega_j} \left\{ \text{tr} \left\{ \sqrt{u_j} \chi_I(H_\omega^{(\Lambda)}) \sqrt{u_j} S \right\} \right\} \leq (\text{tr } S) Q_{\mu_j}(|I|), \quad (1.5)$$

where  $H_\omega^{(\Lambda)}$  denotes the restriction of  $H_\omega$  to  $\Lambda$  with periodic boundary condition. Averaging over all the random variables, the expectation of the number of eigenvalues falling in an interval  $I$  is controlled thanks to the celebrated Wegner estimate [W, CoH, CoHK1, CoHK2]:

$$\mathbb{E} \left\{ \text{tr } \chi_I(H_\omega^{(\Lambda)}) \right\} \leq K_W Q_\Lambda(|I|) |\Lambda|, \quad (1.6)$$

where

$$Q_\Lambda(s) := \max_{j \in \Lambda \cap \mathbb{Z}^d} Q_{\mu_j}(s), \quad (1.7)$$

and the constant  $K_W$  depends on the parameters  $d, u, M_\pm$ , and  $\sup I$ .

In this article we shall investigate the existence of local Wegner estimates of the form

$$\max_{j \in \Lambda \cap \mathbb{Z}^d} \mathbb{E} \left\{ \text{tr } \chi_I(H_\omega^{(\Lambda)}) u_j \right\} \leq K_{LW} Q_\Lambda(|I|). \quad (1.8)$$

If we have a covering condition of the form  $\sum_{j \in \mathbb{Z}^d} u_j \geq C > 0$ , the Wegner estimate (1.6) can be immediately derived from (1.8).

If the random variables  $\{\omega_j\}_{j \in \mathbb{Z}^d}$  are identically distributed, under the above covering condition it is equivalent to investigate local and global Wegner estimates. Indeed, using the covariance property of the model, in this case there exist constants  $C_1$  and  $C_2$  so that for any  $j \in \Lambda$  we have

$$\frac{C_1}{|\Lambda|} \mathbb{E} \left\{ \text{tr } \chi_I(H_\omega^{(\Lambda)}) \right\} \leq \mathbb{E} \left\{ \text{tr } \chi_I(H_\omega^{(\Lambda)}) u_j \right\} \leq \frac{C_2}{|\Lambda|} \mathbb{E} \left\{ \text{tr } \chi_I(H_\omega^{(\Lambda)}) \right\}. \quad (1.9)$$

Our main motivation for introducing this new concept of local Wegner estimates is to prove Minami's estimate for continuous Anderson Hamiltonians. To go beyond the results of [CoGK1], we use local Wegner estimates for Anderson Hamiltonians where the single-site random variables are not identically distributed, i.e., for generalized Anderson Hamiltonians as in Definition 2.1.

## 2. MAIN RESULTS

We write

$$\Lambda_L(x) := x + \left[-\frac{L}{2}, \frac{L}{2}\right]^d \quad (2.1)$$

for the (half open-half closed) box of side  $L > 0$  centered at  $x \in \mathbb{R}^d$ . By  $\Lambda_L$  we denote a box  $\Lambda_L(x)$  for some  $x \in \mathbb{R}^d$ . Given a box  $\Lambda = \Lambda_L(x)$ , we set  $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$ . If  $B$  is a set, we write  $\chi_B$  for its characteristic function. We set  $\chi_x^{(L)} := \chi_{\Lambda_L(x)}$ , with  $\chi_x := \chi_x^{(1)}$ . The Lebesgue measure of a Borel set  $B \subset \mathbb{R}$  will be denoted by  $|B|$ . By a constant we will always mean a finite constant. Constants such as  $C_{a,b,\dots}$  will be finite and depending only on the parameters or quantities  $a, b, \dots$ ; they will be independent of other parameters or quantities in the equation. Note that  $C_{a,b,\dots}$  may stand for different constants in different sides of the same inequality.

Before stating our results, we normalize a generalized Anderson Hamiltonian  $H_\omega$  as follows. We first require  $\inf_{j \in \mathbb{Z}^d} \inf \text{supp } \mu_j = 0$ , which can always be realized by changing the periodic potential  $V_{\text{per}}$ . Second, we set  $\|u\|_\infty = 1$ , which can be achieved by rescaling the  $\mu_j$ . We then adjust  $V_{\text{per}}$  by adding a constant so  $\inf \sigma(-\Delta + V_{\text{per}}) = 0$ , in which case  $[0, E_*] \subset \sigma(-\Delta + V_{\text{per}})$  for some  $E_* > 0$ . The result is a normalized generalized Anderson Hamiltonian as in the following definition, equal to the original generalized Anderson Hamiltonian given in (1.1)-(1.2) plus a nonrandom constant. (In this paper the single site probability distributions have no atoms.)

**Definition 2.1.** *A normalized generalized Anderson Hamiltonian is a generalized Anderson Hamiltonian  $H_\omega$  as in (1.1)-(1.2), such that:*

- (i) *The free Hamiltonian  $H_0 := -\Delta + V_{\text{per}}$  has 0 as the bottom of its spectrum:*

$$\inf \sigma(H_0) = 0. \quad (2.2)$$

- (ii) *The single site potential  $u$  is a measurable function on  $\mathbb{R}^d$  with*

$$\|u\|_\infty = 1 \quad \text{and} \quad u - \chi_{\Lambda_{\delta_-}(0)} \leq u \leq \chi_{\Lambda_{\delta_+}(0)}, \quad \text{where } u_-, \delta_\pm \in ]0, \infty[; \quad (2.3)$$

*we set*

$$U_+ := \left\| \sum_{j \in \mathbb{Z}^d} u_j \right\|_\infty \leq \max \{1, \delta_+^d\}. \quad (2.4)$$

- (iii)  *$\omega = \{\omega_j\}_{j \in \mathbb{Z}^d}$  is a family of independent random variables, such that for all  $j \in \mathbb{Z}^d$  the probability distribution  $\mu_j$  of  $\omega_j$  has no atoms and*

$$0 = \inf_{j \in \mathbb{Z}^d} \inf \text{supp } \mu_j < M := \sup_{j \in \mathbb{Z}^d} \sup \text{supp } \mu_j < \infty. \quad (2.5)$$

*If the  $\{\omega_j\}_{j \in \mathbb{Z}^d}$  are identically distributed, i.e.,  $\mu_j = \mu$  for all  $j \in \mathbb{Z}^d$ ,  $H_\omega$  is a normalized Anderson Hamiltonian, in which case  $\mu$  is a probability measure with no atoms such that*

$$\{0, M\} \in \text{supp } \mu \subset [0, M], \quad \text{where } M \in ]0, \infty[. \quad (2.6)$$

Without loss of generality, we will always assume that a generalized Anderson Hamiltonian  $H_\omega$  is a normalized generalized Anderson Hamiltonian. In particular, Anderson Hamiltonians will also be understood to be normalized.

We will need generalized Anderson Hamiltonian's with more structure. We set

$$\Gamma(j_0, K) := j_0 + K\mathbb{Z}^d, \quad \text{where } j_0 \in \mathbb{Z}^d \quad \text{and} \quad K \in \mathbb{N}. \quad (2.7)$$

Note that for any  $j \in \mathbb{Z}^d$  there exists  $j' \in \Gamma(j_0, K)$  such that  $j \notin \Gamma(j', 2K) \subset \Gamma(j_0, K)$ .

**Definition 2.2.** *A generalized Anderson Hamiltonian  $H_\omega$  has a spine if there exist  $j_0 \in \mathbb{Z}^d$  and  $K \in \mathbb{N}$  such that the random variables  $\{\omega_j\}_{j \in \Gamma(j_0, K)}$  are identically distributed. In this case we will call  $\Gamma = \Gamma(j_0, K)$  a spine of order  $K$  for  $H_\omega$  and set  $\mu_\Gamma := \mu_j$  for  $j \in \Gamma$ .*

An Anderson Hamiltonian  $H_\omega$  (in this language a generalized Anderson Hamiltonian with a spine of order 1) is a  $q\mathbb{Z}^d$ -ergodic family of random self-adjoint operators. It follows from standard results (cf. [KM1]) that there exists fixed subsets  $\Sigma$ ,  $\Sigma_{\text{pp}}$ ,  $\Sigma_{\text{ac}}$  and  $\Sigma_{\text{sc}}$  of  $\mathbb{R}$  so that the spectrum  $\sigma(H_\omega)$  of  $H_\omega$ , as well as its pure point, absolutely continuous, and singular continuous components, are equal to these fixed sets with probability one. With our normalization, the non-random spectrum  $\Sigma$  of an Anderson Hamiltonian  $H_\omega$  satisfies (cf. [KM2])

$$\sigma(H_0) \subset \Sigma \subset [0, \infty[, \quad (2.8)$$

with  $\inf \Sigma = 0$  and  $[0, E_*] \subset \Sigma$  for some  $E_* = E_*(V_{\text{per}}) > 0$ . Note that  $\Sigma = \sigma(-\Delta) = [0, \infty[$  if  $V_{\text{per}} = 0$ .

A generalized Anderson Hamiltonian  $H_\omega$  is not, in general, an ergodic family of random self-adjoint operators, so the above considerations do not apply, and its spectrum is a random set. But it follows from Definition 2.1 that

$$\sigma(H_\omega) \subset [0, \infty[ \quad \text{with probability one.} \quad (2.9)$$

Note furthermore that if the generalized Anderson Hamiltonian  $H_\omega$  has a spine  $\Gamma$  of order  $K$ , then

$$H_{\omega_\Gamma} = H_0 + V_{\omega_\Gamma} \quad \text{where} \quad \omega_\Gamma = \{\omega_j\}_{j \in \Gamma} \quad \text{and} \quad V_{\omega_\Gamma}(x) := \sum_{j \in \Gamma} \omega_j u_j(x), \quad (2.10)$$

is a  $qK\mathbb{Z}^d$ -ergodic family of random self-adjoint operators, and the above considerations for Anderson Hamiltonians apply. ( $H_{\omega_\Gamma}$  is exactly like an Anderson Hamiltonian, except that the single site potentials are located in  $K\mathbb{Z}^d$  instead of  $\mathbb{Z}^d$ .)

Let  $H_\omega$  be a generalized Anderson Hamiltonian. Finite volume operators are defined for finite boxes  $\Lambda = \Lambda_L(j_0)$ , where  $j_0 \in \mathbb{Z}^d$  and  $L \in 2q\mathbb{N}$ ,  $L > \delta_+$ . Given such  $\Lambda$ , we will consider the random Schrödinger operator  $H_\omega^{(\Lambda)}$  on  $L^2(\Lambda)$  given by the restriction of the generalized Anderson Hamiltonian  $H_\omega$  to  $\Lambda$  with periodic boundary condition. To do so, we identify  $\Lambda$  with a torus in the usual way by identifying opposite edges, and define finite volume operators

$$H_\omega^{(\Lambda)} := H_0^{(\Lambda)} + V_\omega^{(\Lambda)} \quad \text{on} \quad L^2(\Lambda). \quad (2.11)$$

The finite volume free Hamiltonian  $H_0^{(\Lambda)}$  is given by

$$H_0^{(\Lambda)} := -\Delta^{(\Lambda)} + V_{\text{per}}^{(\Lambda)} \quad \text{on} \quad L^2(\Lambda), \quad (2.12)$$

where  $\Delta^{(\Lambda)}$  is the Laplacian on  $\Lambda$  with periodic boundary condition and  $V_{\text{per}}^{(\Lambda)}$  is the restriction of  $V_{\text{per}}$  to  $\Lambda$ . The random potential  $V_\omega^{(\Lambda)}$  is the restriction of  $V_\omega$  to  $\Lambda$ , where, given  $\omega = \{\omega_i\}_{i \in \mathbb{Z}^d}$ , we define  $\omega^{(\Lambda)} = \{\omega_i^{(\Lambda)}\}_{i \in \mathbb{Z}^d}$  by

$$\omega_i^{(\Lambda)} = \omega_i \quad \text{if} \quad i \in \tilde{\Lambda}, \quad \omega_k^{(\Lambda)} = \omega_k^{(\Lambda)} \quad \text{if} \quad k - i \in L\mathbb{Z}^d. \quad (2.13)$$

Note that the random finite volume operator  $H_\omega^{(\Lambda)}$  is not covariant with respect to translations in the torus unless  $H_\omega$  is an Anderson Hamiltonian.

Given  $j \in \tilde{\Lambda}$ , we set

$$u_j^{(\Lambda)}(x) := \sum_{k \in j + L\mathbb{Z}^d} u_k(x) \quad \text{and} \quad \chi_j^{(\Lambda)}(x) := \sum_{k \in j + L\mathbb{Z}^d} \chi_k(x) \quad \text{for } x \in \Lambda, \quad (2.14)$$

and rewrite  $V_{\omega}^{(\Lambda)}$  as

$$V_{\omega}^{(\Lambda)} = \sum_{j \in \tilde{\Lambda}} \omega_j u_j^{(\Lambda)}. \quad (2.15)$$

We will often abuse the notation and just write  $u_j$  and  $\chi_j$  instead of  $u_j^{(\Lambda)}$  and  $\chi_j^{(\Lambda)}$  when working with finite volume operators. Note that

$$\sum_{j \in \tilde{\Lambda}} \chi_j^{(\Lambda)}(x) = 1 \quad \text{for all } x \in \Lambda. \quad (2.16)$$

When the covering condition  $\delta_- \geq 1$  (see (2.3)) holds, we have

$$\sum_{j \in \tilde{\Lambda}} u_j^{(\Lambda)}(x) \geq u_- \quad \text{for all } x \in \Lambda. \quad (2.17)$$

Given a finite Borel measure  $\nu$  on  $\mathbb{R}$  with no atoms and finite moments, and  $m \geq 1$ , we set (recall (1.7))

$$Q_{\nu}^{(m)}(s) := Q_{\nu^{(m)}}(s), \quad \text{where} \quad d\nu^{(m)}(t) = (1 + |t|^m) d\nu(t). \quad (2.18)$$

In particular, if  $\text{supp } \nu \subset [0, M]$  (cf. (2.5)) we have

$$Q_{\nu}^{(m)}(s) \leq (1 + M)^m Q_{\nu}(s) \quad \text{for } m \geq 1. \quad (2.19)$$

The finite Borel measure  $\nu$  is said to be Hölder continuous of order  $\alpha \in ]0, 1]$  if there exists a constant  $C_{\nu, \alpha}$  such that

$$Q_{\nu}(s) \leq C_{\nu, \alpha} s^{\alpha} \quad \text{for all } s \in [0, 1]. \quad (2.20)$$

If in addition  $\text{supp } \nu \subset [0, M]$ , it follows that  $\nu^{(m)}$  is also Hölder continuous of order  $\alpha$  for all  $m \geq 1$ :

$$Q_{\nu}^{(m)}(s) \leq C_{\nu, \alpha, m} s^{\alpha} \quad \text{with} \quad C_{\nu, \alpha, m} \leq C_{\nu, \alpha} (1 + M)^m. \quad (2.21)$$

If  $\nu$  has a bounded density  $\rho$  (i.e.,  $\alpha = 1$ ) and  $\text{supp } \nu \subset [0, M]$ , then (2.20) holds with  $C_{\nu, 1} = \|\rho\|_{\infty}$ . In this case, for all  $m \geq 1$  the measure  $\nu^{(m)}$  has a bounded density  $\rho^{(m)}(t) = (1 + t^m)\rho(t)$ , and

$$Q_{\nu}^{(m)}(s) \leq \left\| \rho^{(m)} \right\|_{\infty} s \quad \text{with} \quad \left\| \rho^{(m)} \right\|_{\infty} \leq (1 + M)^m \|\rho\|_{\infty}. \quad (2.22)$$

Let  $H_{\omega}$  be a generalized Anderson Hamiltonian. If  $B \subset \mathbb{R}$  is a Borel set, we write  $P_{\omega}^{(\Lambda)}(B) := \chi_B(H_{\omega}^{(\Lambda)})$  and  $P_{\omega}(B) := \chi_B(H_{\omega})$  for the spectral projections. Let  $E_0 > 0$ ,  $I \subset [0, E_0]$  an interval, and consider a box  $\Lambda = \Lambda_L(j_0)$ , where  $L \in 2q\mathbb{N}$ ,  $L > \delta_+$ , and  $j_0 \in \mathbb{Z}^d$ . If  $H_{\omega}$  satisfies the covering condition  $\delta_- \geq 1$  (see (2.3)), we have the Wegner estimate [CoH, CoGK2] (see also (2.28) below)

$$\mathbb{E} \left\{ \text{tr } P_{\omega}^{(\Lambda)}(I) \right\} \leq K_W(E_0) Q_{\Lambda}(|I|) |\Lambda|. \quad (2.23)$$

Without assuming the covering condition, a careful reading of [CoHK2], as in [GKM, Appendix B], gives

$$\mathbb{E} \left\{ \text{tr } P_{\omega}^{(\Lambda)}(I) \right\} \leq K_W(E_0) Q_{\Lambda}^{(m_a)}(|I|) |\Lambda|, \quad (2.24)$$

where

$$m_d = \min \left\{ 2^n; n \in \mathbb{N}, 2^n > \frac{d}{2} \right\}, \quad (2.25)$$

and

$$Q_\Lambda^{(m)}(s) := \max_{j \in \tilde{\Lambda}} Q_{\mu_j}^{(m)}(s) \leq (1+M)^m Q_\Lambda(|I|) \quad \text{for } m \geq 1. \quad (2.26)$$

The constants  $K_W(E_0)$  in (2.23) and (2.24) depend only on  $d, V_{\text{per}}, \delta_+, u_-$ ; they do not depend on the probability distributions  $\mu_j$ .

If the generalized Anderson Hamiltonian  $H_\omega$  has a spine  $\Gamma$  we set  $Q_\Gamma^{(m)} = Q_{\mu_\Gamma}^{(m)}$ .

We now state our local Wegner estimates. We set  $[\lfloor \frac{d}{4} \rfloor] = \lfloor \frac{d}{4} \rfloor + 1$ , the smallest integer  $> \frac{d}{4}$ .

**Theorem 2.3.** *Let  $H_\omega$  be a generalized Anderson Hamiltonian with  $\delta_- \geq 1$ , and let  $\Lambda = \Lambda_L(j_0)$ , where  $L \in 2q\mathbb{N}$ ,  $L > \delta_+$ , and  $j_0 \in \mathbb{Z}^d$ .*

(i) *Given  $E_0 > 0$ , for all intervals  $I \subset [0, E_0]$  we have*

$$\max_{j \in \tilde{\Lambda}} \mathbb{E} \left\{ \text{tr } P_\omega^{(\Lambda)}(I) u_j^{(\Lambda)} \right\} \leq C_{d, \|V_{\text{per}}^-\|, \delta_+} u_-^{-\frac{3}{2}} (1+E_0)^{[\lfloor \frac{d}{4} \rfloor]} (1 + \log(1+E_0)) Q_\Lambda(|I|). \quad (2.27)$$

*As a consequence, we have the Wegner estimate*

$$\mathbb{E} \left\{ \text{tr } P_\omega^{(\Lambda)}(I) \right\} \leq C_{d, \|V_{\text{per}}^-\|, \delta_+} u_-^{-\frac{5}{2}} (1+E_0)^{[\lfloor \frac{d}{4} \rfloor]} (1 + \log(1+E_0)) Q_\Lambda(|I|) |\Lambda|. \quad (2.28)$$

(ii) *Suppose the generalized Anderson Hamiltonian  $H_\omega$  has a spine  $\Gamma$  of order  $K$ . Given  $\eta \in ]0, \frac{d}{2}[$ , there exists  $E_1 = E_1(\eta, d, V_{\text{per}}, \delta_\pm, u_-, \mu_\Gamma, K) > 0$ , such that for all  $E_0 \in ]0, E_1[$ , and intervals  $I \subset [0, E_0]$ , we have*

$$\max_{j \in \tilde{\Lambda}} \mathbb{E} \left\{ \text{tr } P_\omega^{(\Lambda)}(I) u_j^{(\Lambda)} \right\} \leq e^{-E_0^{-\frac{d}{2} + \eta}} Q_\Lambda(|I|), \quad (2.29)$$

*for  $L$  large (how large depending on  $E_0, d, V_{\text{per}}, \delta_\pm, u_-, \mu_\Gamma, K, \eta$ ).*

(iii) *Suppose the generalized Anderson Hamiltonian  $H_\omega$  has a spine  $\Gamma$  of order  $K$  with a Hölder continuous single-site probability distribution  $\mu_\Gamma$  of order  $\alpha$ , and let  $L \in 2qK\mathbb{N}$ . Then there exists  $E_1 = E_1(d, V_{\text{per}}, \delta_\pm, u_-, \mu_\Gamma, K) > 0$ , such that for all  $E_0 \in ]0, E_1[$ , intervals  $I \subset [0, E_0]$ , and  $\eta \in ]0, 1[$ , we have*

$$\max_{j \in \tilde{\Lambda}} \mathbb{E} \left\{ \text{tr } P_\omega^{(\Lambda)}(I) u_j^{(\Lambda)} \right\} \leq C_\eta \left( C_{\mu_\Gamma} \left( 2\alpha E_0 \log \frac{1}{2\alpha E_0 C_{\mu_\Gamma}^\alpha} \right)^\alpha \right)^{1-\eta} Q_\Lambda(|I|) \quad (2.30)$$

$$\leq C_{\eta, \mu_\Gamma} E_0^{\alpha(1-\frac{\eta}{2})} Q_\Lambda(|I|) \quad (2.31)$$

*for  $L$  large (how large depending on  $d, V_{\text{per}}, \delta_\pm, u_-, \mu_\Gamma, K$ ), where  $C_\eta = C_{d, V_{\text{per}}, \delta_\pm, u_-, K, E_1, \eta}$ ,  $C_{\mu_\Gamma} = C_{\mu_\Gamma, \alpha, m_d}$  as in (2.21) with  $m_d$  given in (2.25), and  $C_{\eta, \mu_\Gamma} = C_{d, V_{\text{per}}, \delta_\pm, u_-, K, E_1, \mu_\Gamma, \eta}$ .*

Part (i), namely (2.27), gives a local version of the Wegner estimates (2.23) and (2.26). It is of the form given in (1.8), valid at all energies  $E_0$  with a constant  $K_{LW} = K_{LW}(E_0)$ , but the constant does not get small as  $E_0 \downarrow 0$ . To prove Minami's estimate (Theorem 2.5) we need a local Wegner estimate valid for small  $E_0$  with  $\lim_{E_0 \downarrow 0} K_{LW}(E_0) = 0$ . We provide two such estimates in parts (ii) and (iii). Part (ii) requires less hypotheses, and seems to provide a stronger result. But we believe that the energy interval  $[0, E_1]$  where the estimates hold is bigger in (iii). The proof of (ii) takes advantage of the Lifshitz tails estimate, and is thus valid in an energy interval at the bottom of the spectrum where we have Lifshitz tails.

The proof of (iii) uses dynamical localization estimates, and is valid in the energy interval where we can perform the bootstrap multiscale analysis of [GK1], which in principle is larger than the region of Lifshitz tails. In addition, (2.30), unlike (2.29), shows the explicit dependence of the constant on the single-site probability distribution  $\mu_\Gamma$ . (This is the reason why we state (2.30) in addition to (2.31).) Note that when  $\mu_\Gamma$  has a bounded density  $\rho$ , as in Theorem 2.5, we have (recall (2.22))  $C_{\mu_\Gamma} = \|\rho^{(m_d)}\|_\infty \leq (1 + M)^{m_d} \|\rho\|_\infty$ .

An Anderson Hamiltonian  $H_\omega$  satisfies a Lifshitz tails estimate, which asserts that its integrated density of states (IDS)  $N(E)$  has exponential fall off as the energy  $E$  approaches the bottom of the spectrum. The finite volume operator  $H_\omega^{(\Lambda)}$  has a compact resolvent, and hence its ( $\omega$ -dependent) spectrum consists of isolated eigenvalues with finite multiplicity. We recall that the integrated density of states (IDS) for  $H_\omega$  is given, for a.e.  $E \in \mathbb{R}$ , by  $(\Lambda_L = \Lambda_L(0))$

$$N(E) := \lim_{L \rightarrow \infty} |\Lambda_L|^{-1} \operatorname{tr} \chi_{[-\infty, E]} \left( H_\omega^{(\Lambda_L)} \right) \quad \text{for } \mathbb{P}\text{-a.e. } \omega, \quad (2.32)$$

in the sense that the limit exists and is the same for  $\mathbb{P}$ -a.e.  $\omega$  (cf. [CL, N, PF]). Recalling that with our normalization the bottom of the spectrum is at 0, the IDS satisfies the Lifshitz tails estimate (e.g., [Klo1, Corollary 2.2 and Remark 7.1])

$$\lim_{E \downarrow 0} \frac{\log |\log N(E)|}{\log E} \leq -\frac{d}{2}. \quad (2.33)$$

Equality is actually known to hold in (2.33).

Since  $N(E)$  is an increasing function, it has a derivative  $n(E) := N'(E) \geq 0$  almost everywhere, the density of states. Note that by ergodicity with respect to  $q\mathbb{Z}^d$  we have

$$N(E) = q^{-d} \mathbb{E} \left\{ \operatorname{tr} \chi_0^{(q)} \chi_{[-\infty, E]} (H_\omega) \chi_0^{(q)} \right\}, \quad (2.34)$$

and hence

$$N(E') - N(E) \leq q^{-d} \mathbb{E} \left\{ \operatorname{tr} \chi_0^{(q)} \chi_{[E, E']} (H_\omega) \chi_0^{(q)} \right\} \quad \text{for } E \leq E'. \quad (2.35)$$

As a consequence, if the single-site probability distribution  $\mu$  has a bounded density  $\rho$ , and the local Wegner estimate (1.8) holds for intervals  $I \subset [0, E_0]$ , we conclude that

$$n(E) \leq q^{-d} K_{LW} \|\rho\|_\infty \quad \text{for a.e. } E \in [0, E_0]. \quad (2.36)$$

The following corollary, which provides an exponentially small bound for the density of states within the regime of Lifshitz tails, is an immediate corollary of Theorem 2.3(ii), using (2.29) and (2.36).

**Corollary 2.4.** *Let  $H_\omega$  be an Anderson Hamiltonian with  $\delta_- \geq 1$ , whose single-site probability distribution  $\mu$  has a bounded density  $\rho$ . Then there exists a Borel set  $\mathcal{N} \subset [0, 1]$  of zero Lebesgue measure such that*

$$\lim_{E \downarrow 0; E \notin \mathcal{N}} \frac{\log |\log n(E)|}{\log E} \leq -\frac{d}{2}. \quad (2.37)$$

The same Lifshitz tails estimate for the density of states holds for the discrete Anderson model [CoGK3].

We now turn to Minami's estimate, originally proved for the discrete Anderson model in [M], and by the authors for the continuous Anderson Hamiltonian

[CoGK2]. The following theorem removes the requirements of a uniform-like probability distribution and a double covering condition of [CoGK2, Theorem 2.2], improving substantially on our previous result

**Theorem 2.5.** *Let  $H_\omega$  be an Anderson Hamiltonian with  $\delta_- \geq 1$ , whose single-site probability distribution  $\mu$  has a bounded density  $\rho$ . Then, there exists  $E_0 = E_0(d, V_{\text{per}}, \delta_\pm, u_-, \mu) > 0$ , such that for all intervals  $I \subset [0, E_0]$  and boxes  $\Lambda = \Lambda_L(j_0)$  with  $j_0 \in \mathbb{Z}^d$  and  $L \in 2q\mathbb{N}$  sufficiently large (how large depending on  $d, V_{\text{per}}, \delta_\pm, u_-, \mu$ ) we have the Minami estimate*

$$\mathbb{E} \left\{ \left( \text{tr } P_\omega^{(\Lambda)}(I) \right) \left( \text{tr } P_\omega^{(\Lambda)}(I) - 1 \right) \right\} \leq K_M (\|\rho\|_\infty |I| |\Lambda|)^2, \quad (2.38)$$

with a constant  $K_M = K_M(d, u, V_{\text{per}}, \delta_+, M, E_0)$ .

Once we have the Minami estimate, in addition to the Wegner estimate and localization estimates, we can investigate statistical properties of the (properly rescaled) finite volume eigenvalues as we consider the thermodynamic limit. As a consequence, thanks to (2.38), we can prove Poisson statistics of eigenvalues and provide the limiting distribution of eigenvalues spacings for an Anderson Hamiltonian  $H_\omega$  with  $\delta_- \geq 1$  whose single-site probability distribution  $\mu$  has a bounded density  $\rho$ , improving on, respectively, [CoGK2] and [GK11]. We shall describe these two applications below, in the particular case where the density of states exists and is strictly positive.

We recall that such an Anderson Hamiltonian  $H_\omega$  exhibits Anderson and dynamical localization at the bottom of the spectrum. More precisely, there exists an energy  $E_1 > 0$  such that  $[0, E_1] \subset \Xi^{\text{CL}}$ , where  $\Xi^{\text{CL}}$  is the region of complete localization for the random operator  $H_\omega$ , defined as the set of energies  $E \in \mathbb{R}$  where we have the conclusions of the bootstrap multiscale analysis of [GK1], which include Anderson and dynamical localization. (See [CoGK2, Appendix A] for a discussion of localization. Note that  $\mathbb{R} \setminus \Sigma \subset \Xi^{\text{CL}}$  in our definition.)

The integrated density of states (IDS)  $N(E)$  for  $H_\omega$  is a nondecreasing absolutely continuous function on  $\mathbb{R}$ , the cumulative distribution function of the density of states measure, given by (recall (2.34))

$$\eta(B) := q^{-d} \mathbb{E} \text{tr} \left\{ \chi_0^{(q)} P_\omega(B) \chi_0^{(q)} \right\} \quad \text{for a Borel set } B \subset \mathbb{R}. \quad (2.39)$$

In particular  $N(E)$  is differentiable a.e. with respect to Lebesgue measure, with  $n(E) := N'(E) \geq 0$  being the density of the measure  $\eta$ , so  $n(E) > 0$  for  $\eta$ -a.e.  $E$ . We let  $\mathcal{E}$  denote the set of energies  $E$  such that  $n(E) = N'(E)$  exists and

$$\lim_{|t|+|s| \rightarrow 0} \frac{N(E+t) - N(E+s)}{t-s} = n(E). \quad (2.40)$$

The set  $\mathcal{E}$  (obviously) contains the continuity points of  $n(E)$  and is of full Lebesgue measure [GK11].

Given an energy  $E \in \Sigma$  where IDS is differentiable at  $E$  with  $n(E) := N'(E) > 0$ , we define a point process  $\xi_{E,\omega}^{(\Lambda)}$  on the real line by the rescaled spectrum of the finite volume operator  $H_\omega^{(\Lambda)}$  near  $E$ :

$$\begin{aligned} \xi_{E,\omega}^{(\Lambda)}(B) &:= \# \{j \in \mathbb{N}, n(E) |\Lambda| (E_j(\omega, \Lambda) - E) \in B\} \\ &= \text{tr} \left\{ \chi_B \left( n(E) |\Lambda| \left( H_\omega^{(\Lambda)} - E \right) \right) \right\} = \text{tr} \left\{ P_\omega^{(\Lambda)} \left( E + (n(E) |\Lambda|)^{-1} B \right) \right\} \end{aligned} \quad (2.41)$$



for a Borel set  $B \subset \mathbb{R}$ , where  $\{E_j(\omega, \Lambda)\}_{j \in \mathbb{N}}$  are the eigenvalues of the finite volume operator  $H_\omega^{(\Lambda)}$ .

For sake of simplicity, we restrict ourselves to the particular case where the density of states exists and is strictly positive (see the remark below Theorem 2.6 for extensions).

**Theorem 2.6.** *Let  $H_\omega$  be an Anderson Hamiltonian satisfying with  $\delta_- \geq 1$ , whose single-site probability distribution  $\mu$  has a bounded density  $\rho$ . Then there exists an energy  $E_0 > 0$ , with  $[0, E_0[ \subset \Xi^{CL}$ , such that:*

- (i) *With probability one, every eigenvalue of  $H_\omega$  in  $[0, E_0[$  is simple.*
- (ii) *For all energies  $E \in [0, E_0[$  such that the IDS is differentiable at  $E$  with  $n(E) := N'(E) > 0$ , the point process  $\xi_{E, \omega}^{(\Lambda_L)}$  converges weakly, as  $L \rightarrow \infty$ , to the Poisson point process  $\xi_E$  on  $\mathbb{R}$  with intensity measure  $\nu_E(B) := \mathbb{E} \xi_E(B) = |B|$ , i.e.,  $\nu_E$  is the Lebesgue measure on the real line.*
- (iii) *Fix  $E \in [0, E_0[ \cap \mathcal{E}$  with  $n(E) > 0$ , let  $\{I_\Lambda\}_\Lambda$  be a sequence of subintervals of  $[0, E_0[$  such that  $\lim_{\Lambda \rightarrow \mathbb{R}^d} \sup I_\Lambda = 0$ . Assume that  $\lim_{\Lambda \rightarrow \mathbb{R}^d} |\Lambda|^{1-\delta} |I_\Lambda| = \infty$  for some  $\delta > 0$ , and that  $\lim_{\Lambda \rightarrow \mathbb{R}^d} \frac{|I_{\Lambda_L + \ell'}|}{|I_{\Lambda_L}|} = 1$  if  $\ell' = o(L)$ . Then, with probability one, as  $\Lambda \rightarrow \mathbb{R}^d$ , the empirical distribution function*

$$\frac{\#\{j \in \mathbb{N}; E_j(\omega, \Lambda) \in E + I_\Lambda, n(E) |\Lambda| (E_{j+1}(\omega, \Lambda) - E_j(\omega, \Lambda)) \geq t\}}{N(I_\Lambda) |\Lambda|} \quad (2.42)$$

*converges uniformly to the exponential distribution  $e^{-t}$ .*

Theorem 2.6(i) and (ii) improve on [CoGK2, Theorem 2.1]; there is no requirement of a double covering condition and a uniform-like distribution. They are immediate consequences of Theorem 2.5 and [CoGK2, Theorem 2.3]. (Note that (i) follows immediately from Minami's estimate as shown in [KIM].) Theorem 2.6(iii) follows immediately from Theorem 2.5 and [GK11, Theorem 1.5].

Statistics of eigenvalues can still be investigated when the density of states  $n(E)$  does not exist or is zero. In this case, one rescales eigenvalues through the IDS and study the so-called unfolded eigenvalues. In practice, it consists in replacing the quantities  $n(E)(E_j(\omega, \Lambda) - E)$  by  $(N(E_j(\omega, \Lambda)) - N(E))$  in (2.41) and  $n(E)(E_{j+1}(\omega, \Lambda) - E_j(\omega, \Lambda))$  by  $N(E_{j+1}(\omega, \Lambda)) - N(E_j(\omega, \Lambda))$  in (2.42), see [GK11, K102, GK12].

In addition to Theorem 2.6, one may also use the Minami estimate of Theorem 2.5 to investigate the joint eigenvalues/centers of localization statistics as in [GK11], the asymptotic eigenvalue ergodicity as in [K102], as well as statistics at spectral edges as in [GK12]. Higher order Minami estimates as in [CoGK1, Theorem 2.3] (discrete case) should also hold for continuous models, but the proof still seems to be out of reach.

### 3. PROOF OF LOCAL WEGNER ESTIMATES

#### 3.1. A simple Lemma.

**Lemma 3.1.** *Let  $H = H_0 + W$ , where  $H, H_0$  are semi-bounded self-adjoint operators, say  $H, H_0 \geq -\Theta$  for some  $\Theta > 0$ , such that  $(H + \Theta + 1)^{-p}$  is a trace class operator for some  $p > 0$ , and  $W$  is a bounded self-adjoint operator. Let  $E_0 \in \mathbb{R}$ . Let  $f, h$  be bounded Borel measurable nonnegative functions with compact support*

such that  $f = \chi_{(-\infty, E_0]} f$ ,  $h = \chi_{[E_0, \infty)} h$ , and  $H_0 h(H_0)$  is a bounded operator. Then  $f(H)Wh(H_0)$  is trace class and

$$\operatorname{tr} f(H)Wh(H_0) \leq 0. \quad (3.1)$$

In particular, if  $f, g$  are bounded Borel measurable nonnegative functions such that  $f = \chi_{(-\infty, E_0]} f$  and  $\chi_{(-\infty, E_0]} \leq g \leq 1$ , we have  $f(H)W$  and  $f(H)Wg(H_0)$  trace class, and

$$\operatorname{tr} f(H)W \leq \operatorname{tr} f(H)Wg(H_0). \quad (3.2)$$

Note that  $W$  does not need to be positive.

*Proof.* Let  $f, h$  be as above, note that  $f(H)$  is trace class. Then, as  $W = H - H_0$ , we have

$$\operatorname{tr} f(H)Wh(H_0) = \operatorname{tr} f(H)Hh(H_0) - \operatorname{tr} f(H)H_0h(H_0), \quad (3.3)$$

where both  $f(H)Hh(H_0)$  and  $f(H)H_0h(H_0)$  are trace class operators. Moreover,

$$\operatorname{tr} f(H)Hh(H_0) \leq E_0 \operatorname{tr} f(H)h(H_0), \quad (3.4)$$

$$\operatorname{tr} f(H)H_0h(H_0) \geq E_0 \operatorname{tr} f(H)h(H_0), \quad (3.5)$$

so (3.1) follows.

Now let  $f, g$  be as above. Let also  $\chi_n = \chi_{(-\infty, n]}$ . Then, using (3.1),

$$\begin{aligned} \operatorname{tr} f(H)W &= \lim_{n \rightarrow \infty} \operatorname{tr} f(H)W\chi_n(H_0) \\ &= \operatorname{tr} f(H)Wg(H_0) + \lim_{n \rightarrow \infty} \operatorname{tr} f(H)W\chi_n(H_0)(1 - g(H_0)) \\ &\leq \operatorname{tr} f(H)Wg(H_0). \end{aligned} \quad (3.6) \quad \square$$

**3.2. Norms on random operators.** Given  $p \in [1, \infty)$ ,  $\mathcal{T}_p$  will denote the Banach space of bounded operators  $S$  on  $L^2(\mathbb{R}^d, dx)$  with  $\|S\|_{\mathcal{T}_p} = \|S\|_p := (\operatorname{tr} |S|^p)^{\frac{1}{p}} < \infty$ . A random operator  $S_\omega$  is a strongly measurable map from the probability space  $(\Omega, \mathbb{P})$  to bounded operators on  $L^2(\mathbb{R}^d, dx)$ . Given  $p \in [1, \infty)$ , we set

$$\| \| S_\omega \| \|_p := \{ \mathbb{E} \{ \| S_\omega \|_p^p \} \}^{\frac{1}{p}} = \| \| S_\omega \| \|_{\mathcal{T}_p(\Omega, \mathbb{P})}, \quad (3.7)$$

and

$$\| \| S_\omega \| \|_\infty := \| \| S_\omega \| \|_{L^\infty(\Omega, \mathbb{P})}. \quad (3.8)$$

These are norms on random operators, note that

$$\| \| S_\omega \| \|_q \leq \| \| S_\omega \| \|_\infty^{\frac{q-p}{q}} \| \| S_\omega \| \|_p^{\frac{p}{q}} \quad \text{for } 1 \leq p \leq q < \infty, \quad (3.9)$$

and they satisfy Holder's inequality:

$$\| \| S_\omega T_\omega \| \|_r \leq \| \| S_\omega \| \|_p \| \| T_\omega \| \|_q \quad \text{for } r, p, q \in [1, \infty] \text{ with } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \quad (3.10)$$

### 3.3. Proof of Theorem 2.3.

*Proof.* Let  $H_\omega$  be a generalized Anderson Hamiltonian satisfying the covering condition  $\delta_- \geq 1$  (see (2.3)), and consider a box  $\Lambda = \Lambda_L$ , where  $L \in 2q\mathbb{N}$ ,  $L > \delta_+$ . Let  $E_0 > 0$ ,  $I \subset [0, E_0]$  an interval. Let  $g$  be a bounded Borel measurable function such that  $\chi_{(-\infty, E_0]} \leq g \leq 1$ . Given  $j \in \mathbb{Z}^d$ , we let  $\omega_j^\perp = \{\omega_k\}_{k \in \mathbb{Z}^d \setminus \{j\}}$ , write  $\omega = (\omega_j^\perp, \omega_j)$ , and consider the random Schrödinger operator  $H_{\omega_j^\perp} = H_\omega - \omega_j u_j$ .

To simplify the notation, we will write  $u_k$  and  $\chi_k$  for  $u_k^{(\Lambda)}$  and  $\chi_k^{(\Lambda)}$ , and set

$$\hat{\chi}_k := u_k^{-\frac{1}{2}} \chi_k \leq u_-^{-\frac{1}{2}} \chi_k \quad \text{for } k \in \tilde{\Lambda} \quad (\text{recall (2.3) and } \delta_- \geq 1), \quad (3.11)$$

$$P = P_{\omega}^{(\Lambda)}(I) := \chi_I(H_{\omega}^{(\Lambda)}), \quad (3.12)$$

$$\tilde{P}_j = \tilde{P}_{j, \omega_j^\perp}^{(\Lambda)}(I) := g(H_{\omega_j^\perp}^{(\Lambda)}), \quad \text{where } j \in \tilde{\Lambda} \quad \text{and} \quad H_{\omega_j^\perp}^{(\Lambda)} = H_{\omega}^{(\Lambda)} - \omega_j u_j. \quad (3.13)$$

Given  $j \in \tilde{\Lambda}$ , it follows from Lemma 3.1, using (2.16), that

$$\begin{aligned} \text{tr } P u_j &\leq \text{tr } P u_j \tilde{P}_j = \sum_{k \in \tilde{\Lambda}} \text{tr } P u_j \tilde{P}_j \chi_k = \sum_{k \in \tilde{\Lambda}} \text{tr } u_k^{\frac{1}{2}} P u_j^{\frac{1}{2}} u_j^{\frac{1}{2}} \tilde{P}_j \hat{\chi}_k \\ &= \sum_{k \in \tilde{\Lambda}} \text{tr } u_k^{\frac{1}{2}} P u_j^{\frac{1}{2}} T_{j,k}, \end{aligned} \quad (3.14)$$

where

$$T_{j,k} = u_j^{\frac{1}{2}} \tilde{P}_j \hat{\chi}_k. \quad (3.15)$$

It follows that

$$\begin{aligned} \mathbb{E} \{ \text{tr } P u_j \} &= \left\| \left\| P u_j^{\frac{1}{2}} \right\|_2 \right\|^2 \leq \sum_{k \in \tilde{\Lambda}} \left\| \left\| u_k^{\frac{1}{2}} P u_j^{\frac{1}{2}} T_{j,k} \right\|_1 \right\|_1 \\ &\leq \sum_{k \in \tilde{\Lambda}} \left\| \left\| u_k^{\frac{1}{2}} P \right\|_2 \right\| \left\| \left\| P u_j^{\frac{1}{2}} T_{j,k} \right\|_2 \right\| \leq \left( \max_{r \in \tilde{\Lambda}} \left\| \left\| P u_r^{\frac{1}{2}} \right\|_2 \right\| \right) \sum_{k \in \tilde{\Lambda}} \left\| \left\| P u_j^{\frac{1}{2}} T_{j,k} \right\|_2 \right\|, \end{aligned} \quad (3.16)$$

and hence

$$\max_{r \in \tilde{\Lambda}} \left\| \left\| P u_r^{\frac{1}{2}} \right\|_2 \right\| \leq \max_{j \in \tilde{\Lambda}} \sum_{k \in \tilde{\Lambda}} \left\| \left\| P u_j^{\frac{1}{2}} T_{j,k} \right\|_2 \right\|. \quad (3.17)$$

We have

$$\begin{aligned} \left\| \left\| P u_j^{\frac{1}{2}} T_{j,k} \right\|_2 \right\|^2 &= \mathbb{E} \left\{ \text{tr} \left\{ P u_j^{\frac{1}{2}} T_{j,k} T_{j,k}^* u_j^{\frac{1}{2}} P \right\} \right\} = \mathbb{E} \left\{ \text{tr} \left\{ u_j^{\frac{1}{2}} P u_j^{\frac{1}{2}} T_{j,k} T_{j,k}^* \right\} \right\} \\ &\leq Q_{\mu_j}(|I|) \mathbb{E}_{\omega_j^\perp} \left\{ \text{tr } T_{j,k} T_{j,k}^* \right\} = Q_{\mu_j}(|I|) \left\| \left\| T_{j,k} \right\|_2 \right\|^2, \end{aligned} \quad (3.18)$$

where we used the basic spectral averaging estimate (1.5) (note that  $T_{j,k}$  does not depend on  $\omega_j$ ). It follows that

$$\max_{r \in \tilde{\Lambda}} \mathbb{E} \{ \text{tr } P u_r \} \leq Q_{\Lambda}(|I|) \left( \max_{j \in \tilde{\Lambda}} \sum_{k \in \tilde{\Lambda}} \left\| \left\| T_{j,k} \right\|_2 \right\| \right)^2. \quad (3.19)$$

To prove (i), we use (3.9) with

$$\left\| \left\| T_{j,k} \right\|_1 \right\| \leq u_-^{-1} \left\| \left\| \tilde{P}_j u_j^{\frac{1}{2}} \right\|_2 \right\| \left\| \left\| \tilde{P}_j u_k^{\frac{1}{2}} \right\|_2 \right\| \leq u_-^{-1} \max_{r \in \tilde{\Lambda}} \mathbb{E} \left\{ \text{tr } \tilde{P}_j u_r \right\}, \quad (3.20)$$

to conclude that

$$\max_{r \in \tilde{\Lambda}} \mathbb{E} \{ \text{tr } P u_r \} \leq u_-^{-1} Q_{\Lambda}(|I|) \left( \max_{j,k \in \tilde{\Lambda}} \mathbb{E} \left\{ \text{tr } \tilde{P}_j u_k \right\} \right) \left( \max_{j \in \tilde{\Lambda}} \sum_{k \in \tilde{\Lambda}} \left\| \left\| T_{j,k} \right\|_2 \right\| \right)^{\frac{1}{2}}. \quad (3.21)$$

If the function  $g$  in (3.13) satisfies  $g(E) = 0$  for  $E > E_1 \geq E_0$ , it follows from the usual trace estimate for Schrödinger operators (e.g., [GK3, Lemma A.4]) that

$$\mathrm{tr} \tilde{P}_j u_k \leq C_{d, \|V_{\mathrm{per}}^-\|, \delta_+} (1 + E_1)^{[\frac{d}{4}]} \quad \text{for all } j, k \in \tilde{\Lambda} \quad \text{and } \omega \in [0, \infty[^{\mathbb{Z}^d}, \quad (3.22)$$

where  $V_{\mathrm{per}}^-$  denotes the negative part of  $V_{\mathrm{per}}$  and  $[\frac{d}{4}]$  is the smallest integer  $> \frac{d}{4}$ . We now take  $g(E) = g_0(E - E_0)$ , where  $g_0 \in C^\infty(\mathbb{R})$ ,  $0 \leq g_0 \leq 1$ ,  $g_0(E) = 1$  for  $E \leq 0$ , and  $g_0(E) = 0$  for  $E \geq 1$ . We now apply [GK2, Theorem 2], concluding that for all  $n \in \mathbb{N}$ ,  $j, k \in \tilde{\Lambda}$ , and  $\omega \in [0, \infty[^{\mathbb{Z}^d}$  we have

$$\|T_{j,k}\| \leq u_-^{-\frac{1}{2}} \left\| u_j^{\frac{1}{2}} \tilde{P}_j \chi_k \right\| \leq u_-^{-\frac{1}{2}} \left\| \chi_{\Lambda_{\delta_+}(j)} \tilde{P}_j \chi_k \right\| \leq C_{d, \|V_{\mathrm{per}}^-\|, \delta_+, n} u_-^{-\frac{1}{2}} \frac{1 + \log(1 + E_0)}{(1 + d_\Lambda(j, k))^n}, \quad (3.23)$$

where  $d_\Lambda(\cdot, \cdot)$  is the distance on the torus  $\Lambda = \Lambda_L$ :

$$d_\Lambda(y, y') = \min_{r \in L\mathbb{Z}^d} |y - y' + r| \quad \text{for } y, y' \in \Lambda. \quad (3.24)$$

(Note that the results in [GK2] are valid on the torus with the appropriate modifications, the main one being the use of the distance on the torus.) Taking  $n = 2d + 2$ , and using

$$\sum_{k \in \tilde{\Lambda}} (1 + d_\Lambda(j, k))^{-(d+1)} \leq \sum_{k \in \mathbb{Z}^d} (1 + |k|)^{-(d+1)} < \infty \quad \text{for all } j \in \tilde{\Lambda}, \quad (3.25)$$

we conclude that

$$\left( \max_{j \in \tilde{\Lambda}} \sum_{k \in \tilde{\Lambda}} \| \|T_{j,k}\| \|_\infty^{\frac{1}{2}} \right)^2 \leq C_{d, \|V_{\mathrm{per}}^-\|, \delta_+} u_-^{-\frac{1}{2}} (1 + \log(1 + E_0)). \quad (3.26)$$

It now follows from (3.21), (3.22), and (3.26) that

$$\max_{r \in \tilde{\Lambda}} \mathbb{E} \{ \mathrm{tr} P u_r \} \leq C_{d, \|V_{\mathrm{per}}^-\|, \delta_+} (1 + E_0)^{[\frac{d}{4}]} (1 + \log(1 + E_0)) u_-^{-\frac{3}{2}} Q_\Lambda(|I|), \quad (3.27)$$

which is (2.27). The Wegner estimate (2.28) is an immediate consequence of (2.27) and (2.17). This finishes the proof of (i).

Now suppose that the generalized Anderson Hamiltonian  $H_\omega$  has a spine  $\Gamma$  of order  $K$ . For any  $j \in \mathbb{Z}^d$  there exists a spine  $\Gamma_j \subset \Gamma$  of order  $2K$  with  $j \notin \Gamma_j$ , and we can write

$$H_{\omega_j^\perp} = H_{\omega_{\Gamma_j}} + V_{\omega_j^\perp \setminus \omega_{\Gamma_j}}, \quad \text{where } 0 \leq V_{\omega_j^\perp \setminus \omega_{\Gamma_j}} := V_{\omega_j^\perp} - V_{\omega_{\Gamma_j}} \leq U_+. \quad (3.28)$$

We take the function  $g$  in (3.13) so  $0 \leq g \leq 1$ ,  $g(E) = 1$  for  $E \leq E_0$ ,  $g(E) = 0$  for  $E \geq E^*$ ; where  $E^* \geq E_0$  will be later chosen appropriately. We have (writing  $H_{\omega_j^\perp}$  for  $H_{\omega_j^\perp}^{(\Lambda)}$ , etc.)

$$\mathrm{tr} \left\{ \tilde{P}_j u_k \right\} \leq e^{tE^*} \mathrm{tr} \left\{ e^{-tH_{\omega_j^\perp}} u_k \right\} \leq e^{tE^*} \mathrm{tr} \left\{ e^{-tH_{\omega_{\Gamma_j}}} u_k \right\} \quad \text{for } t > 0, \quad (3.29)$$

where we used (3.28) and the positivity preserving property as in [BoGKS, Lemma 2.2]. Setting

$$P_{\omega_{\Gamma_j}}([0, E]) := \chi_{[0, E]}(H_{\omega_{\Gamma_j}}^{(\Lambda)}) = \chi_{]-\infty, E]}(H_{\omega_{\Gamma_j}}^{(\Lambda)}), \quad (3.30)$$

we get, again using the positivity preserving property as in [BoGKS, Lemma 2.2], and requiring  $t \geq 2$ , that for all  $E > 0$  we have

$$\begin{aligned} \operatorname{tr} \left\{ e^{-tH_{\omega_{\Gamma_j}}} u_k \right\} &\leq \operatorname{tr} \left\{ P_{\omega_{\Gamma_j}}([0, E]) u_k \right\} + e^{-\frac{t}{2}E} \operatorname{tr} \left\{ e^{-\frac{t}{2}H_{\omega_{\Gamma_j}}} u_k \right\} \\ &\leq \operatorname{tr} \left\{ P_{\omega_{\Gamma_j}}([0, E]) u_k \right\} + e^{-\frac{t}{2}E} \operatorname{tr} \left\{ e^{-\frac{t}{2}H_0} u_k \right\} \\ &\leq \operatorname{tr} \left\{ P_{\omega_{\Gamma_j}}([0, E]) u_k \right\} + e^{-\frac{t}{2}E} \operatorname{tr} \left\{ e^{-H_0} u_k \right\} \\ &\leq \operatorname{tr} \left\{ P_{\omega_{\Gamma_j}}([0, E]) u_k \right\} + C_{d, V_{\text{per}}, \delta_+} e^{-\frac{t}{2}E}. \end{aligned} \quad (3.31)$$

Since  $\Gamma_j$  is a spine of order  $2K$  and  $L \in 2qK\mathbb{N}$ , the random operator  $H_{\omega_{\Gamma_j}}^{(\Lambda)}$  is covariant in the torus  $\Lambda$ , and we have

$$\begin{aligned} \mathbb{E} \left\{ \operatorname{tr} \left\{ P_{\omega_{\Gamma_j}}([0, E]) u_k \right\} \right\} &= \frac{1}{\#(\Gamma_j \cap \Lambda)} \sum_{r \in j + \Gamma_j \cap \Lambda} \mathbb{E}_{\omega_{\Gamma_j}} \left\{ \operatorname{tr} \left\{ P_{\omega_{\Gamma_j}}([0, E]) u_r \right\} \right\} \\ &\leq \frac{(2K)^d}{|\Lambda|} U_+ \mathbb{E}_{\omega_{\Gamma_j}} \left\{ \operatorname{tr} P_{\omega_{\Gamma_j}}([0, E]) \right\} \end{aligned} \quad (3.32)$$

for all  $E > 0$ . Combining (3.29), (3.31), and (3.32) we get

$$\mathbb{E} \left\{ \operatorname{tr} \left\{ \tilde{P}_j u_k \right\} \right\} \leq C_{d, V_{\text{per}}, \delta_+, K} e^{tE^*} \left( |\Lambda|^{-1} \mathbb{E}_{\omega_{\Gamma_j}} \left\{ \operatorname{tr} P_{\omega_{\Gamma_j}}([0, E]) \right\} + e^{-\frac{t}{2}E} \right). \quad (3.33)$$

To prove (ii), we take  $E_0 \in ]0, \frac{1}{8}]$ , fix  $E^* = 2E_0$ , and require  $g \in C^\infty(\mathbb{R})$  with  $|g^{(j)}(E)| \leq CE_0^{-j}$  for all  $E \in \mathbb{R}$  and  $j = 1, 2, \dots, 2d+4$ , where  $C$  is a constant independent of  $E$ . Applying [GK2, Theorem 2] as in (3.23), we get

$$\|T_{j,k}\| \leq C_{d, \|V_{\text{per}}^-\|, \delta_+} u_-^{-\frac{1}{2}} E_0^{-2d-3} (1 + d_\Lambda(j, k))^{-2d-2}, \quad (3.34)$$

and conclude, similarly to (3.26)

$$\left( \max_{j \in \tilde{\Lambda}} \sum_{k \in \tilde{\Lambda}} \| \|T_{j,k}\| \|_\infty^{\frac{1}{2}} \right)^2 \leq C_{d, \|V_{\text{per}}^-\|, \delta_+} u_-^{-\frac{1}{2}} E_0^{-2d-3}. \quad (3.35)$$

Thus, it follows from (3.21) and (3.35) that

$$\max_{r \in \tilde{\Lambda}} \mathbb{E} \left\{ \operatorname{tr} P u_r \right\} \leq C_{d, \|V_{\text{per}}^-\|, \delta_+} u_-^{-\frac{3}{2}} E_0^{-2d-3} Q_\Lambda(|I|) \left( \max_{j, k \in \tilde{\Lambda}} \mathbb{E} \left\{ \operatorname{tr} \tilde{P}_j u_k \right\} \right). \quad (3.36)$$

Note that  $H_{\omega_{\Gamma_j}}$  would be an Anderson Hamiltonian but for the fact that the random potential is located on  $\Gamma_j$  instead of  $\mathbb{Z}^d$ . All the results for Anderson Hamiltonians apply to  $H_{\omega_{\Gamma_j}}$ , with the obvious modifications.  $H_{\omega_{\Gamma_j}}$  is a  $2qK\mathbb{Z}^d$ -ergodic family of random self-adjoint operators. It has an integrated density of states  $N_{\Gamma_j}(E)$ , defined similarly to (2.32), a continuous function in view of the Wegner estimate (2.24). It follows from (2.32) that for all  $E \in \mathbb{R}$  there exists  $L(E)$  such that for all boxes  $\Lambda = \Lambda_L$  with  $L \geq L(E)$  we have

$$|\Lambda|^{-1} \mathbb{E} \left( \operatorname{tr} \chi_{]-\infty, E]} \left( H_{\omega_{\Gamma_j}}^{(\Lambda)} \right) \right) \leq 2N_{\Gamma_j}(E). \quad (3.37)$$

$N_{\Gamma_j}(E)$  satisfies the Lifshitz tails estimate (2.33), so it follows that given  $\eta \in ]0, \frac{1}{2}]$  there exists  $E^*(\eta) > 0$  such that

$$N(E) \leq e^{-E^{-\frac{d}{2}+\eta}} \quad \text{for all } E \in [0, E^*(\eta)]. \quad (3.38)$$

We conclude that

$$|\Lambda_L|^{-1} \mathbb{E} \left( \text{tr } \chi_{]-\infty, E]} \left( H_{\omega_{\Gamma_j}}^{(\Lambda_L)} \right) \right) \leq 2e^{-E^{-\frac{d}{2}+\eta}} \quad \text{for } E \in [0, E^*(\eta)], \quad L \geq L(E). \quad (3.39)$$

In particular, requiring  $8E_0 \leq E^*(\eta)$  and  $L \geq L(8E_0)$ , it follows from (3.33) with  $E^* = 2E_0$  and  $E = 8E_0$ , (3.37), and (3.38), that

$$\mathbb{E} \left\{ \text{tr } \left\{ \tilde{P}_j u_k \right\} \right\} \leq C_{d, V_{\text{per}}, \delta_+, K} e^{2tE_0} \left( e^{-(8E_0)^{-\frac{d}{2}+\eta}} + e^{-4tE_0} \right). \quad (3.40)$$

We now choose  $t$  by ( $t \geq 2$  since  $E_0 \leq \frac{1}{8}$ )

$$e^{-(8E_0)^{-\frac{d}{2}+\eta}} = e^{-4tE_0}, \quad \text{i.e., } t = \frac{1}{4} (8E_0)^{-1-\frac{d}{2}+\eta}, \quad (3.41)$$

getting

$$\begin{aligned} \mathbb{E} \left\{ \text{tr } \left\{ \tilde{P}_j u_k \right\} \right\} &\leq 2C_{d, V_{\text{per}}, \delta_+, K} e^{2tE_0} e^{-4tE_0} = 2C_{d, V_{\text{per}}, \delta_+, K} e^{-2tE_0} \\ &= 2C_{d, V_{\text{per}}, \delta_+, K} e^{-\frac{1}{2}(8E_0)^{-\frac{d}{2}+\eta}}. \end{aligned} \quad (3.42)$$

Thus, if  $8E_0 \leq E^*(\eta)$  and  $L \geq L(8E_0)$  it follows from (3.36) and (3.42) that

$$\max_{r \in \tilde{\Lambda}} \mathbb{E} \{ \text{tr } P u_r \} \leq C_{d, V_{\text{per}}, \delta_+, K} u^{-\frac{3}{2}} Q_{\Lambda}(|I|) E_0^{-2d-3} e^{-\frac{1}{2}(8E_0)^{-\frac{d}{2}+\eta}}. \quad (3.43)$$

It follows that there is  $E^\dagger(\eta) = E^\dagger(\eta, d, V_{\text{per}}, \delta_+, u_-, K, \mu_{\Gamma}) > 0$  such that for  $E_0 \leq E^\dagger(\eta)$  and  $L \geq L(8E_0)$  we get

$$\max_{r \in \tilde{\Lambda}} \mathbb{E} \{ \text{tr } P u_r \} \leq e^{-E_0^{-\frac{d}{2}+\eta}} Q_{\Lambda}(|I|) \quad \text{for } I \subset [0, E_0], \quad (3.44)$$

which is (2.29). Thus (ii) is proven.

To prove (iii), we also assume that  $\mu_{\Gamma}$  is Hölder continuous, so (2.24) and (2.26) yield a Wegner estimate that allows the performance of the bootstrap multiscale analysis [GK1, Kl] for the random Schrödinger operator  $H_{\omega_{\Gamma_j}}$ , and hence for  $H_{\omega_j^\perp}$  by treating  $V_{\omega_j^\perp \setminus \omega_{\Gamma_j}}$  in (3.28) as a fixed nonnegative uniformly bounded background potential as in [GK4]. The ‘a priori’ finite volume estimate required for starting the multiscale analysis is given by [GK4, Proposition 4.3]. It follows that there exists  $E_1 > 0$  such that we can perform a bootstrap multiscale analysis for  $H_{\omega_j^\perp}$  (using only the random variables  $\omega_{\Gamma_j}$ ), the constants being uniform in  $j \in \mathbb{Z}^d$ . In particular, taking  $0 < E_0 \leq E_1$ ,  $g = \chi_{]-\infty, E_0]}$  (in particular,  $E^* = E_0$ ), so  $\tilde{P}_j = \chi_{]-\infty, E_0]}(H_{\omega_j^\perp})$ , we conclude that (this follows from the multiscale analysis as in [GK1, Kl], see also [R]; the argument holds in finite volume) for  $L$  large (how large depending on  $d, V_{\text{per}}, \delta_{\pm}, u_-, \mu_{\Gamma}, K$ )

$$\|T_{j,k}\|_2 \leq u^{-\frac{1}{2}} \left\| \chi_{\Lambda_{\delta_+}(j)} \tilde{P}_j \chi_k \right\|_2 \leq C_{d, V_{\text{per}}, \delta_{\pm}, u_-} e^{-\sqrt{d_{\Lambda}(j,k)}} \quad \text{for } j, k \in \tilde{\Lambda}. \quad (3.45)$$

In particular, give  $s > 0$ , we have

$$\sum_{k \in \tilde{\Lambda}} e^{-s\sqrt{d_{\Lambda}(j,k)}} \leq \sum_{k \in \mathbb{Z}^d} e^{-s|k|^{\frac{1}{2}}} = C_{d,s} < \infty \quad \text{for all } j \in \tilde{\Lambda}. \quad (3.46)$$

Since we also have

$$\|T_{j,k}\|_2 = \left\| u_j^{\frac{1}{2}} \tilde{P}_j \hat{\chi}_k \right\|_2 \leq u_-^{-\frac{1}{2}} \left\| u_j^{\frac{1}{2}} \tilde{P}_j \right\|_2 = u_-^{-\frac{1}{2}} \left( \mathbb{E} \left\{ \text{tr } \tilde{P}_j u_j \right\} \right)^{\frac{1}{2}}, \quad (3.47)$$

it follows from (3.19), (3.45), (3.46), and (3.47), that for any  $\eta \in ]0, 1[$  we have

$$\max_{r \in \tilde{\Lambda}} \mathbb{E} \left\{ \text{tr } P u_r \right\} \leq C_{d,V_{\text{per}},\delta_{\pm},u_{-},\eta} Q_{\Lambda}(|I|) \left( \max_{j \in \tilde{\Lambda}} \mathbb{E} \left\{ \text{tr } \tilde{P}_j u_j \right\} \right)^{1-\eta}. \quad (3.48)$$

We now consider energies  $0 < E_2 \leq E_3$ ; we will fix  $E_3$  later. It follows from (3.33) with  $E^* = E_0$ ,  $E = E_2$ , and  $t = \frac{1}{E_0}$ , that

$$\mathbb{E} \left\{ \text{tr } \left\{ \tilde{P}_j u_k \right\} \right\} \leq C_{d,V_{\text{per}},\delta_{+},K} \left( |\Lambda|^{-1} \mathbb{E}_{\omega_{\Gamma_j}} \left\{ \text{tr } P_{\omega_{\Gamma_j}}([0, E_2]) \right\} + e^{-\frac{E_2}{2E_0}} \right). \quad (3.49)$$

Using (2.24) and (2.21), we get

$$\mathbb{E} \left\{ \text{tr } \left\{ P_{\omega_{\Gamma_j}}([0, E_2]) \right\} \right\} \leq C_{E_3} Q_{\Gamma}^{(m_d)}(E_2) \leq C_{E_3} C_{\mu_{\Gamma},\alpha,m_d} E_2^{\alpha}, \quad (3.50)$$

the constant  $C_{E_3}$  depending only on  $d, V_{\text{per}}, \delta_{+}, u_{\pm}, K$  and on  $E_3$ . Combining (3.49) and (3.50) we get

$$\mathbb{E} \left\{ \text{tr } \left\{ \tilde{P}_j u_j \right\} \right\} \leq C_1 \left( C_{\mu_{\Gamma}} E_2^{\alpha} + e^{-\frac{E_2}{2E_0}} \right), \quad (3.51)$$

with a constant  $C_1 = C_{d,V_{\text{per}},\delta_{+},u_{\pm},K,E_1,E_3}$  and  $C_{\mu_{\Gamma}} = C_{\mu_{\Gamma},\alpha,m_d}$ .

Let  $\beta(s)$  be defined on  $[0, \infty[$  by  $\beta(0) = 0$  and

$$C_{\mu_{\Gamma}} (\beta(s))^{\alpha} = e^{-\frac{\beta(s)}{2s}} \quad \text{for } s > 0. \quad (3.52)$$

In particular,

$$C_{\mu_{\Gamma}} (\beta(s))^{\alpha} e^{\frac{\beta(s)}{2s}} = 1, \quad \text{i.e.,} \quad \frac{\beta(s)}{2\alpha s} e^{\frac{\beta(s)}{2\alpha s}} = \left( 2\alpha s C_{\mu_{\Gamma}}^{\frac{1}{\alpha}} \right)^{-1}. \quad (3.53)$$

If

$$\left( 2\alpha s C_{\mu_{\Gamma}}^{\frac{1}{\alpha}} \right)^{-1} \geq 3, \quad \text{i.e.,} \quad 6\alpha s C_{\mu_{\Gamma}}^{\frac{1}{\alpha}} \leq 1, \quad (3.54)$$

we have

$$\frac{\beta(s)}{2\alpha s} \leq \log \left( 2\alpha s C_{\mu_{\Gamma}}^{\frac{1}{\alpha}} \right)^{-1}, \quad \text{i.e.,} \quad \beta(s) \leq 2\alpha s \log \left( 2\alpha s C_{\mu_{\Gamma}}^{\frac{1}{\alpha}} \right)^{-1}. \quad (3.55)$$

We now choose  $E_2 = \beta(E_0)$  and  $E_3 = \beta(E_1)$ , and require

$$E_0 \leq \tilde{E}_1 = \min \left\{ E_1, \left( 6\alpha C_{\mu_{\Gamma}}^{\frac{1}{\alpha}} \right)^{-1} \right\}. \quad (3.56)$$

It follows from (3.51) and (3.55) that

$$\mathbb{E} \left\{ \text{tr } \left\{ \tilde{P}_j u_j \right\} \right\} \leq 2C_1 C_{\mu_{\Gamma}} \left( 2\alpha E_0 \log \frac{1}{2\alpha E_0 C_{\mu_{\Gamma}}^{\frac{1}{\alpha}}} \right)^{\alpha}. \quad (3.57)$$

The estimate (2.30) follows immediately from (3.48) and (3.57). This proves (iii).  $\square$

## 4. PROOF OF MINAMI'S ESTIMATE

*Proof of Theorem 2.5.* We start with a preliminary remark. Given  $\delta > 0$  small, we pick a nonincreasing function  $h \in C^\infty(\mathbb{R})$ , such that  $h(t) = 1$  for  $t \leq 0$  and  $h(t) = 0$  for  $t \geq \delta$ . Note that  $0 \leq h \leq 1$ ,  $h' \leq 0$ ,  $\text{supp } h' \subset [0, \delta]$ ,  $\int_{\mathbb{R}} dt h'(t) = -1$ , and we can choose  $h$  so  $|h'| \leq \frac{2}{\delta}$ . Given  $c \in \mathbb{R}$ , we set  $h_c(t) = h(t - c)$ , and note that  $h_{c-\delta} \leq \chi_{]-\infty, c]} \leq h_c$ . Let us consider  $I = ]a, b] \subset [0, \infty[$ , and write  $I_\delta = ]a - \delta, b + \delta]$ . As shown in [CoGK2, Eqs. (5.3)-(5.4)], for all  $j \in \tilde{\Lambda}$  we have

$$\text{tr } P_{\omega}^{(\Lambda)}(I) \leq \xi_{b,\tau}^{(\Lambda)}(\omega_j^\perp) + \text{tr } P_{(\omega_j^\perp, \tau)}^{(\Lambda)}(I_\delta) \quad (4.1)$$

where, for any  $\tau \geq M$  (recall (2.6)),

$$\begin{aligned} \xi_{b,\tau}^{(\Lambda)}(\omega_j^\perp) &:= \text{tr } h_b \left( H_{(\omega_j^\perp, \omega_j=0)}^{(\Lambda)} \right) - \text{tr } h_b \left( H_{(\omega_j^\perp, \omega_j=\tau)}^{(\Lambda)} \right) \\ &= - \int_0^\tau ds \text{tr} \left\{ \sqrt{u_j} h'_b \left( H_{(\omega_j^\perp, \omega_j=s)}^{(\Lambda)} \right) \sqrt{u_j} \right\} \\ &\leq \frac{2}{\delta} \int_0^\tau ds \text{tr} \left\{ \sqrt{u_j} P_{(\omega_j^\perp, \omega_j=s)}^{(\Lambda)}(|b, b + \delta|) \sqrt{u_j} \right\}. \end{aligned} \quad (4.2)$$

It follows that for all  $\tau \geq M$  we have

$$\mathbb{E}_{\omega_j^\perp} \left\{ \xi_{b,\tau}^{(\Lambda)}(\omega_j^\perp) \right\} \leq \frac{2}{\delta} \tau \mathbb{E}_{\omega'} \text{tr} \left\{ u_j P_{\omega'}^{(\Lambda)}(|b, b + \delta|) \right\}, \quad (4.3)$$

where  $\omega' = \{\omega'_j\}_{j \in \mathbb{Z}^d}$  are independent random variables such that  $\omega'_k = \omega_k$  for  $k \neq j$ , and  $\omega'_j$  is uniformly distributed on the interval  $[0, \tau]$ . Noting that  $H_{\omega'}$  is a generalized Anderson Hamiltonian with a spine  $\Gamma$  of order 2 and  $\mu_\Gamma = \mu$ , we apply Theorem 2.3(iii) to conclude that there exists  $E_1 > 0$  (independent of  $\tau$ ), such that if  $b + \delta \leq E_0 \leq E_1$  we have (note  $1 \leq M \|\rho\|_\infty$  and recall (2.22) )

$$\begin{aligned} \mathbb{E}_{\omega_j^\perp} \left\{ \xi_{b,\tau}^{(\Lambda)}(\omega_j^\perp) \right\} &\leq C \frac{2\tau}{\delta} \left( 2 \left\| \rho^{(m_d)} \right\|_\infty E_0 \log \frac{1}{2 \left\| \rho^{(m_d)} \right\|_\infty E_0} \right)^{\frac{1}{2}} \left( \max \left\{ \|\rho\|_\infty, \frac{1}{\tau} \right\} \right) \delta \\ &\leq 2C\tau \|\rho\|_\infty \left( 2 \left\| \rho^{(m_d)} \right\|_\infty E_0 \log \frac{1}{2 \left\| \rho^{(m_d)} \right\|_\infty E_0} \right)^{\frac{1}{2}}, \end{aligned} \quad (4.4)$$

where the constant  $C$  does not depend on either  $E_0$ ,  $\tau$ , or  $\mu$ . Thus, we can choose  $E_0 \in ]0, E_1]$  such that if  $b + \delta \leq E_0$  we have

$$\mathbb{E}_{\omega_j^\perp} \left\{ \xi_{b,\tau}^{(\Lambda)}(\omega_j^\perp) \right\} \leq 1 \quad \text{for all } \tau \in [M, 2M]. \quad (4.5)$$

Note also that it follows from Theorem 2.3(i), and recalling that  $1/M \leq \|\rho\|_\infty$ , that

$$\frac{1}{M} \int_M^{2M} \mathbb{E}_{\omega_j^\perp} \left\{ \text{tr } P_{(\omega_j^\perp, \tau)}^{(\Lambda)}(I_\delta) \right\} d\tau \leq C \|\rho\|_\infty |I_\delta| |\Lambda|, \quad (4.6)$$

where  $C = C_{d, \|\mathbf{V}_{\text{per}}^-\|, \delta_+, u_-, E_0}$ .

Now let us fix  $E_0 > 0$  as above, and consider  $I = ]a, b] \subset [0, E_0[$ . We recall that it follows from [CoH, CoHK2], as discussed in [CoGK2, Eqs. (4.1)-(4.5)], that

$$\text{tr } P_{\omega}^{(\Lambda)}(I) \leq Q_1 \sum_{j \in \tilde{\Lambda}} \text{tr} \left\{ \sqrt{u_j} P_{\omega}^{(\Lambda)}(I) \sqrt{u_j} S_j^{(\Lambda)} \right\}, \quad (4.7)$$



with

$$\max_{j \in \tilde{\Lambda}} \left\{ \operatorname{tr} S_j^{(\Lambda)} \right\} \leq Q_2, \quad (4.8)$$

where  $Q_1, Q_2$  are constants depending only on  $E_0, d, u, V_{\text{per}}, M$ . If  $\operatorname{tr} P_{\omega}^{(\Lambda)}(I) \geq 1$  and  $\delta > 0$ , it follows that

$$\left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) \right) \left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) - 1 \right) \leq Q_1 \sum_{j \in \tilde{\Lambda}} \operatorname{tr} \left\{ \sqrt{u_j} P_{\omega}^{(\Lambda)}(I) \sqrt{u_j} S_j^{(\Lambda)} \right\} \left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) - 1 \right). \quad (4.9)$$

Thus, using (4.1), we get

$$\left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) \right) \left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) - 1 \right) \leq Q_1 \sum_{j \in \tilde{\Lambda}} \left\{ \left( \operatorname{tr} \left\{ \sqrt{u_j} P_{\omega}^{(\Lambda)}(I) \sqrt{u_j} S_j^{(\Lambda)} \right\} \right) \Phi_{b,\tau}^{(\Lambda)}(\omega_j^{\perp}) \right\} \quad (4.10)$$

for all  $\tau \geq M$ , where for each  $j \in \tilde{\Lambda}$

$$\Phi_{b,\tau}^{(\Lambda)}(\omega_j^{\perp}) := \left( \xi_{b,\tau}^{(\Lambda)}(\omega_j^{\perp}) - 1 \right) + \operatorname{tr} P_{(\omega_j^{\perp}, \tau)}^{(\Lambda)}(I_{\delta}) \quad (4.11)$$

is independent of the random variable  $\omega_j$ . If  $\operatorname{tr} P_{\omega}^{(\Lambda)}(I) < 1$ , we have  $P_{\omega}^{(\Lambda)}(I) = 0$ , and hence we also have (4.10).

Thus, if  $\tau \in [M, 2M]$ , taking the expectation in (4.10), using (1.5), (4.8), and (4.5), we get

$$\begin{aligned} \mathbb{E} \left\{ \left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) \right) \left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) - 1 \right) \right\} &\leq Q_1 Q_2 \|\rho\|_{\infty} |I| \sum_{j \in \tilde{\Lambda}} \mathbb{E}_{\omega_j^{\perp}} \left\{ \Phi_{b,\tau}^{(\Lambda)}(\omega_j^{\perp}) \right\} \quad (4.12) \\ &\leq Q_1 Q_2 \|\rho\|_{\infty} |I| \sum_{j \in \tilde{\Lambda}} \mathbb{E}_{\omega_j^{\perp}} \left\{ \operatorname{tr} P_{(\omega_j^{\perp}, \tau)}^{(\Lambda)}(I_{\delta}) \right\}. \end{aligned}$$

Since this holds for all  $\tau \in [M, 2M]$ , it follows from (4.6) that

$$\begin{aligned} \mathbb{E} \left\{ \left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) \right) \left( \operatorname{tr} P_{\omega}^{(\Lambda)}(I) - 1 \right) \right\} &\leq \lim_{\delta \downarrow 0} Q_1 Q_2 \|\rho\|_{\infty} |I| |\Lambda| (C \|\rho\|_{\infty} |I_{\delta}| |\Lambda|) \\ &= K_M (\|\rho\|_{\infty} |I| |\Lambda|)^2, \quad (4.13) \end{aligned}$$

where  $K_M = C Q_1 Q_2$  depends on  $d, u, V_{\text{per}}, \delta_+, M, E_0$ .  $\square$

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